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NOTES ON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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Abstract : In this paper we obtain a common fixed point theorem for six self maps on fuzzy metric space by using implicit relation.

Keywords : Common fixed points, fuzzy metric space, weak compatible maps, compatible maps of type(β), implicit relation.

1 INTRODUCTION

THE The concept of fuzzy sets was introduced initially by Zadeh [13]. Subsequently, several resercheral defined fuzzy metric space in various methods in various abstract spaces. In this paper, We deal with the fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [3]. They obtained that every metric space induces a fuzzy metric spaces. Also Grebic [4] has proved fixed point theorem for fuzzy metric space. In fuzzy metric space, the notion of compatible maps was initiated by Mishra, Sharma and Singh [8]. Recently Jungek and Rhoades [5] introduced and worked on the concept of weak compatible maps.

In this paper we prove a fixed point theorem for A, B, S, T, P and Q self maps using implicit relation. Concept of weak compatibility and compatibility of type (β) of self maps are characterized to get common fixed points.

2 PRELIMINARIES

Definition 2.1 [13] : A binary operation *, [0, 1] × [0, 1] → [0, 1] is called at norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that

$$a*b \le c*d$$

Whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$

Definition 2.2 [12] : The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$. Satisfying the following conditions :

For all $x,y,z \in X$ and $s,t > 0$

(FM 1) $M(x,y,0) = 0$

(FM 2) $M(x,y,t) = 1$ for all $f > 0$ f if and only if $x = y$

(FM 3) $M(x,y,t) = M(y,x,t)$

(FM 4) $M(x,y,t) * M(y,z,s) \le M(x+z,t+s)$

(FM 5) $M(x,y,.):(0,\infty) \to [0,1]$ is left continuous

(FM 6) $\lim_{n\to\infty} M(x,y,t) = 1$

Note that every metric spaces induces a Fuzzy metric space.

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Definition 2.3 [11] Let (X, d) be a Metric Space. Define $a*b = \min\{a,b\}$ and $M(x,y,t) = t/[t + d(x,y)]$ for all $x,y \in X$ and $t > 0$.

Then $(X,M,*)$ is a fuzzy metric space. It is called the fuzzy metric space induced by d.

Definition 2.4 [3] : A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is defined as Cauchy sequence if for each $\in > 0$ and $t > 0$, there exists $x \in N$ such that $M(x_n, x_m, t) > 1-\in$ for all $n,m \ge x_0$ the sequence $\{x_n\}$ is said to be converge to a point x in X iff

$$M(x_n, x, t) > 1-\in \text{ for all } n,m \ge x_0$$

A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.5 [1] : A pair of self mappings (A, S) of fuzzy metric $(X,M,*)$ is said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) \to 1 \forall t > 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = x \text{ for some } x \in X$$

If the self mapping A and B of a fuzzy metric space $(X,M,*)$ are compatible then they are weakly compatible but its converse is not true.

Lemma 2.1 [9] : Let $(X,M,*)$ be a fuzzy metric space, then for all $x,y \in X, M(x,y,.)$ is non-decreasing.

Lemma 2.2 [10] : Let $(X,M,*)$ be a fuzzy metric space. If there exists $k \in (0,1)$ such that $M(x,y,kt) \ge M(x,y,t) \forall t > 0$ and $x,y \in X$ then $x = y$

Proof : Let there exists, $k \in (0,1)$ such that $M(x,y,kt) \ge M(x,y,t)$ for all $x,y \in X$ $t > 0$ then $M(x,y,kt) \ge M(x,y,t/k)$ and so $M(x,y,kt) \ge M(x,y,t/k^n)$ for positive integer n, taking limit $n \to \infty$ $M(x,y,k) \ge 1$ and hence $x = y$

Lemma 2.3 [12] : The only t-norm and satisfying $r*r \ge r$ for all $r \in [0,1]$ is the minimum t-norm, that is

$a*b = \min\{a,b\}$ for all $a,b \in [0,1]$

Bijendra singh et al.[11] have proved the following result:

Let $(X,M,*)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mapping from X, into itself such that the following conditions are satisfied :

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(a) $P(X) \subset ST(X), Q(X) \subset AB)(X$

(b) either AB or P is continuous

(c) (P, AB) is compatible and (Q, ST) is weakly compatible.

(d) AB = BA, ST = TS, PB = BP, QT = TQ

(e) There exist $k \in (0,1)$ such that for every $x, y \in X, t > 0$

$$M(Px, Qy, qt) \geq M(ABx, STy, t) * M(Px, ABx, t)$$
$$* M(Qy, STy, t) * M(Px, STy, t)$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Now we are introducing an implicit-relation and we will prove the above result by using implicit relation in place of contractive condition.

A Class of Implicit Relation:

Let ϕ be the set of all real continuous functions $\phi(t_1, t_2, t_3, t_4, t_5) : (R+)^5 \to R$ which is non-decreasing in the Ist argument satisfying the following conditions :

$(\phi 1) \to$ for $u, v \geq 0$

$\phi(u, v, v, u, 1) \geq 0$ implies $u \geq v$

$(\phi 2) \to$ $\phi(v, u, 1, 1, u) \geq 0$ or $\phi(u, 1, u, 1, u) \geq 0$

Or $\phi(u, 1, 1, u, 1) \geq 0$ implies $u \geq 1$

Example : $\phi(t_1, t_2, t_3, t_4, t_5) = 13t_1 - 11t_2 - 3t_3 - 5t_4 + t_5 - 1$

3 Main Result

Theorem 3.1 : Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mapping from X into itself such that the following conditions are satisfied :

(a) $P(X) \subset ST(X), Q(X) \subset AB(X)$

(b) Either A B or P is continuous.

(c) (P, AB) is compatible of type β and (Q, ST) is weakly compatible

(d) AB = BA, ST = TS, PB = BP, QT = TR

(e) there exist $k \in (0,1)$ such that for every $x, y \notin X$ and $t > 0$

$$\phi(M(Px, Qy, kt), M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, kt)$$
$$M(Px, STy, t) \geq 0$$

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof – Let $x_0 \in X$ from 3.1(a) there exist $x_1, x_2 \in X$ such that

$Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_2 = y_1$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X, such that

$Px_{2n} = STx_{2n+1} = y_{2n}$ and

$Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ….

Step 1 : We put in contractive condition $x = x_{2n}, y = x_{2n+1}$ then we get

$$\phi(M(Px_{2n}, Qx_{2n+1}, kt), M(ABx_{2n}, STx_{2n+1}, t),$$
$$M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, kt), M(Px_{2n}, STx_{2n+1}, t) \geq 0$$
$$\Rightarrow \phi(M(y_{2n}, y_{2n+1}, kt), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t)$$
$$M(y_{2n+1}, y_{2n}, kt), M(y_{2n}, y_{2n}, t) \geq 0$$

$\Rightarrow$ From property of implicit relation

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$$
$$M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$$

Similarly we have

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t)$$

Therefore, for all n, we have

$$M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$$

To prove that $\{y_n\}$ is a Cauchy sequence,

$$M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t/k) \geq M(y_{n-1}, y_{n-2}, t/k^2)$$
$$\geq \ldots \geq M(y, y_o, t/k^n) \to 1$$

as $n \to \infty$

Thus, the result hold for m = 1

By induction hypothesis suppose that result hold for m = p,

$$M(y_n, y_{n-p+1}, t) \geq M(y_n, y_{n-p}, t/2) * M(y_{n+1}, y_{n-p+1}, t/2) \to 1*1 = 1$$

Thus the result holds for $m = p + 1$.

Hence $\{y_n\}$ is a Cauchy sequence in X. Which is complete therefore $\{y_n\}$ convergence to z i.e. $y_n \to z \in X$. Also its sub-sequences converges to the same point i.e. $z \in X$.

$$\{Qx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z$$
$$\{Px_{2n}\} \to z \text{ and } \{ABx_{2n}\} \to z$$

Case 1 : Suppose AB is continuous.

As AB is continuous,

$(AB)^2 x_{2n} \to ABz$ and

$(AB)Px_{2n} \to ABz$

As (P, AB) is compatible pair, we have

$P(AB)x_{2n} \to ABz$

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Step 2 : Putting $x = ABx_{2n}, y = x_{2n+1}$ in contractive condition, we have

$$\phi(M(PABx_{2n}, Qx_{2n+1}, kt), M(ABABx_{2n}, STx_{2n+1}, t), ,$$
$$M(PABx_{2n}, ABABx_{2n}, t), M(Qx_{2n}, STx_{2n+1}, kt),$$
$$M(PABx_{2n}, STx_{2n+1}, t)) \geq 0$$

Letting $n \to \infty$, we get

$$\phi(M(ABz, z, kt), M(ABz, z, t), M(ABz, ABz, t),$$
$$M(z, z, t), \mathrm{M(ABz,z,t)} \;) \geq 0$$
$$\phi(M(ABz, z, kt), M(ABz, z, t), 1, 1, M(ABz, z, t)) \geq 0$$

Since ϕ is non-decreasing in first arguments so,

$$\phi(M(ABz, z, t), M(ABz, z, t), 1, 1, M(ABz, z, t) \geq 0$$

By the property of implicit relation

$$M(ABz, z, t) \geq 1$$

$$\Rightarrow \quad ABz = z$$

Step 3 : Putting $x = z, y = x_{2n+1}$, in contractive condition,

we get

$$\phi(M(Pz, Qx_{2n+1}, kt), M(ABz, STx_{2n+1}, t), M(Pz, ABz, t),$$
$$M(Qx_{2n+1}, STx_{2n+1}, kt), M(Pz, STx_{2n+1}, t)) \geq 0$$

Letting Limit $n \to \infty$, we get

$$\phi(M(Pz, z, kt), M(z, z, t), M(Pz, z, t), M(z, z, t), M(Pz, z, t) \geq 0$$

Since ϕ is non-decreasing in first argument, so

$$\phi(M(Pz, z, t), 1, M(Pz, z, t), 1, M(Pz, z, t)) \geq 0$$

By the property of implicit relation

$$M(Pz, z, t) \geq 1$$

$$\Rightarrow \quad Pz = z$$

Therefore $\quad ABz = Pz = z$

Step 4 : Put $x = Bz$ and $y = x_{2n+1}$ in contractive condition we get

$$\phi(M(PBz, Qx_{2n+1}, kt), M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t),$$
$$M(Qx_{2n+1}, STx_{2n+1}, kt), M(PBz, STx_{2n+1}, t)) \geq 0$$

$\because$ As BP = PB, AB = BA so we have

P (Bz) = B (Pz) = Bz and

AB (Bz) = BA (Bz) = B (ABz) = Bz]

Letting limit $n \to \infty$, we get

$$\phi(M(Bz, z, kt), M(Bz, z, t), M(Bz, Bz, t), M(z, z, kt), M(Bz, z, t)) \geq 0$$

Since ϕ is non decreasing in first argument, so we can write

$$\phi(M(Bz, z, t), M(Bz, z, t), 1, 1, M(Bz, z, t)) \geq 1$$

By property of implicit relation

$$\Rightarrow \quad M(Bz, z, t) \geq 1$$

$$\Rightarrow \quad Bz = z$$

Now $\quad Bz = z, ABz = z \Rightarrow Az = z$

therefore $Az = Bz = Pz = z$

Step 5 : As $P(X) \subset ST(X)$, therefore exist $v \in X$. Such that $z = Pz = STv$.

Putting $x = x_{2n}, y = v$, we get from contractive condition, we get,

$$\phi(M(Px_{2n}, Qv, kt), M(ABx_{2n}, STv, t), M(Px_{2n}, ABx_{2n}, t)$$
$$M(Qv, STv, kt), M(Px_{2n}, STv, t) \geq 0$$

Letting $n \to \infty$ and using $\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$ we get

$$\phi(M(z, Qv, kt), M(z, z, t), M(z, z, t), M(Qv, z, kt), M(z, z, t)) \geq 0$$

$$\Rightarrow \quad \phi(M(z, Qv, t), 1, 1, M(z, Qv, kt) \geq 0$$

$$\Rightarrow \quad \phi(M(z, Qv, t) \geq 0$$

$$\Rightarrow \quad Qv = z$$

$$\because \quad STv = z = Qv$$

As (Q, ST) is weakly compatible, we have $STQv = QSTv$.

Thus $STz = Qz$.

Step 6 : Put $x = x_{2n}$, $y = z$ in contractive condition, we get

$$\phi(M(Px_{2n}, Qz, kt), M(ABx_{2n}, STz, t), M(Px_{2n}, ABx_{2n}, t),$$
$$M(Qz, STz, kt), M(Px_{2n}, STz, t) \geq 0$$

Letting $n \to \infty$ and using equation

$$\{Qx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z$$

And Step 5, we get

$$\phi(M(z, Qz, kt), M(z, z, t), M(z, z, t), M(Qz, z, kt), M(z, z, t) \geq 0$$

$$\Rightarrow \quad \phi(M(z, Qz, kt), 1, 1, M(z, Qz, kt), 1) \geq 0$$

$\Rightarrow \quad M(z, Qz, t) \geq 1$ (Using property of implicit relation)

$$\Rightarrow \quad Qz = z$$

Step 7 : Put $x = x_{2n}$, $y = Tz$ in contractive condition we get

$$\phi(M(Px_{2n}, QTz, kt), M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t),$$
$$M(QTz, STTz, kt), M(Px_{2n}, STTz, t) \geq 0$$

As QT = TQ and ST = TS, we have

QTz = TQz = Tz and ST (Tz) = T (STz) = Tz

Letting $n \to \infty$,w e get

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$$\phi(M(z,Tz,kt),M(z,Tz,t),M(z,z,t),M(Tz,Tz,kt),M(z,Tz,t))\geq 0$$

$$\phi(M(z,Tz,kt),M(z,Tz,t),1,1,M(z,Tz,t))\geq 0$$

Since ϕ is non-decreasing in first argument so

$$\phi(M(z,Tz,t),M(z,Tz,t),1,1,M(z,Tz,t))\geq 0$$

$\Rightarrow$ $M(z,Tz,t)\geq 1$ (Using property of implicit relation)

$\Rightarrow$ $Tz=z$

Now $STz=Tz=z\Rightarrow Sz=z$ hence

$$Sz=Tz=Qz=z$$

Combining result, we get

$$Az=Bz=Pz=Qz=Tz=Sz=z$$

Case II : Suppose P is continuous

As P is continuous, $P^2x_{2n}\to Pz$ and $P(AB)x_{2n}\to Pz$

As (P, AB) is compatible of type (ß)

$$M(PPx^n,(AB)(AB)x^n,t)\to 1$$

$$M(Pz,(AB)^2x^n,t)\to 1 \quad \text{when } t>0$$

we have $(AB)^2x_{2n}\to Pz$

Step 8 : Put $x=Px_{2n},y=x_{2n+1}$.

$$\phi(M(PPx_{2n},Qx_{2n+1},kt),M(ABPx_{2n},STx_{2n+1},t),$$
$$M(PPx_{2n},ABPx_{2n},t),M(Qx_{2n+1},STx_{2n+1},kt),M(PPx_{2n},STx_{2n+1},t))\geq 0$$

Letting $n\to\infty$, we get

$$\phi(M(Pz,z,kt),M(Pz,z,t),M(Pz,Pz,t),M(z,z,kt),M(pz,z,t))\geq 0$$

$\Rightarrow$ $\phi(M(Pz,z,kt),1,M(Pz,z,t),1,1,M(Pz,z,t)\geq 0$

Since ϕ is non decreasing in the first argument so we can write

$\Rightarrow$ $\phi(M(Pz,z,t),M(Pz,z,t),1,1,M(Pz,z,t)\geq 0$

$\Rightarrow$ $M(Pz,z,t)\geq 1$ (By property of implicit relation)

$\Rightarrow$ $Pz=z$

Further using Step 5, 6, 7 we get

$$Qz=STz=Sz=Tz=z$$

Step 9 : As $Q(X)\subset AB(X)$, there exists $w\in X$ such that $z=Qz=ABw$ put $x=w,y=x_{2n+1}$ in contractive condition we get

$$\phi(M(Pw,Qx_{2n+1},kt),M(ABw,STx_{2n+1},t),M(Pw,ABw,t),$$
$$M(Qx_{2n+1},STx_{2n+1},kt),M(Pw,STx_{2n+1},t)\geq 0$$

Letting $n\to\infty$ we get

$$\phi(M(Pw,z,kt),M(z,z,t),M(Pw,z,t),M(z,z,kt),M(Pw,z,t))\geq 0$$

$\Rightarrow$ $\phi(M(Pw,z,kt),1,M(Pw,z,t),1,M(Pw,z,t))\geq 0$

Since ϕ is non decreasing in the first argument

$$\phi(M(Pw,z,t),1,M(Pw,z,t),1,M(Pw,z,t)\geq 0$$

$\Rightarrow$ $M(Pw,z,t)\geq 1$ (Using property of implicit relation)

$\Rightarrow$ $Pw=z=Abw$

As (P, AB) is compatible, we have

$$Pz=ABz$$

Also from Step 4, we get Bz = z

Thus Az = Bz = Pz = z and we due that z is the common fixed point of the six maps in the case also.

Uniqueness – Let u be another common fixed point of A, B, S, T, P and Q

Then Au = Bu = Pu = Qu = Su = Tu = u

Put $x=z,y=u$ in contractive condition, we get

$$\phi(M(Pz,Qu,kt),M(ABz,STu,t),M(Pz,ABz,t),$$
$$M(Qu,STu,kt),M(Pz,STu,t)\geq 0$$

Taking $n\to\infty$

$$\phi(M(z,u,kt),M(z,u,t),M(z,z,t),M(u,u,kt),M(z,u,t)\geq 0$$

$\Rightarrow$ $\phi(M(z,u,kt),M(z,u,t),1,1,M(z,u,t)\geq 0$

Since ϕ is non decreasing in first argument so

$$\phi(M(z,u,t),M(z,u,t),1,1,M(z,u,t)\geq 0$$

$\Rightarrow$ $M(z,u,t)\geq 1$ (Using property of implicit relation)

$\Rightarrow$ $z=u$

Therefore z is the unique common fixed point of day maps A, B, S, T, P and Q.

Remark 3.1: If we take B = T = 1, the identify maps on x in the theorem 3.1 ,the condition (b) is satisfied trivially.

Corollary 3.1 : Let $(X,M,*)$ be a complete fuzzy metric space and let A, S, P and Q be mapping from X into itself such that the following conditions are satisfied :

(a) $P(X)\subset S(X),Q(X)\subset A(X)$

(b) Either A or P is continuous.

(c) (P, A) is compatible of type β and (Q, S) is weakly compatible

(d) there exist $k\in(0,1)$ such that for every $x,y\notin X$ and $t>0$

$$\phi\big(\;M(Px,Qy,kt),M(Ax,Sy,t),M(Px,Ax,t),(Qy,Sy,kt)$$
$$M(Px,Sy,t)\;\big)\geq 0$$

Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2 Corollary 3.1 is generalization of the result of cho [

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] in the sense that condion of compatibility of the pairs of self maps ,has been restricted to compatibility of type (β) and weak compatibility and only one map of the pair is needed to continuous.

4 Conclusion

We establish A Common Fixed Point Theorem in Fuzzy Metric Spaces satisfying implicit relations for weakly compatible maps. There are some possible application in engineering, economics in dealing with problems arising in approximation theory, information system. In future scope we can obtain new implicit relation to relax conditions.

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